12th Week Wednesday

10 Mayıs 2023 Çarşamba 14:30

$$A_{B,C} = \left[[L(b_1)]_C \quad [L(b_2)]_C \cdots \right]$$

$A_{B,C}$ → I get the coordinates of input vector with respect to the basis B and I give the coordinates of the output of this vector with respect to the basis C.

$L : V \to W$

standart $\leftarrow v \longmapsto L(v) \to$ standart

$A \longrightarrow$ standart representation matrix

$[v]_B \longrightarrow [L(v)]_C$

$\boxed{A_{B,C}} [v]_B = [L(v)]_C$

$v \longmapsto L(v)$

$\boxed{A} v = L(v)$

Ex $L : \mathbb{R}^2 \to \mathbb{R}^2$

$x b_1 + y b_2 \longmapsto (x+y)c_1 + (x-y)c_2$

$B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{b_2} \right\}$ $C = \left\{ \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{c_1}, \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{c_2} \right\}$

$A_{E,E} = ?$ The standart representation matrix

$L: x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 4 \end{bmatrix} \longmapsto (x+y)\begin{bmatrix} -1 \\ 0 \end{bmatrix} + (x-y)\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$A = \left[L(e_1) \quad L(e_2) \right]$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$L: (x+3y, 2x+4y) \longmapsto (2x-4y, 2x-2y)$

$L(e_1) = L((1,0)) = (-8, -6)$

$\left.\begin{matrix} x+3y = 1 \\ 2x+4y = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} y = 1 \\ x = -2 \end{matrix}$ $2.(-2) - 4.1$ $2.(-2) - 2.1$

$L(e_2) = L((0,1)) = (5, 4)$

$\left.\begin{matrix} x+3y = 0 \\ 2x+4y = 1 \end{matrix}\right\} \Rightarrow \begin{matrix} y = -1/2 \\ x = 3/2 \end{matrix}$ $2.\frac{3}{2} - 4.\frac{-1}{2}$ $2.\frac{3}{2} - 2.\frac{-1}{2}$

$$A = \begin{bmatrix} -8 & 5 \\ -6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 5 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8x+5y \\ -6x+4y \end{bmatrix}$$

$L: (x,y) \longmapsto (-8x+5y, -6x+4y)$

$B^{-1} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \cdot \frac{1}{-2} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$

Ex $L : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (-8x+5y, -6x+4y)$

$B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{b_2} \right\}$ $C = \left\{ \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{c_1}, \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{c_2} \right\}$ -2

$-6.3 + 4.4$

Find the rep. matrix $A_{B,C} = ?$

$\to L(b_1) = L((1,2)) = (2,2) \longrightarrow$ I need the coordinates of this wrt the basis C.

$$A_{B,C} = \left[[L(b_1)]_C \quad [L(b_2)]_C \right]$$

the co... of this wrt the basis C.

$$A_{B,C} = \begin{bmatrix} [L(b_1)]_C & [L(b_2)]_C \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 2 & -3 \\ 0 & -1 \end{bmatrix} \cdot \frac{1}{-2} = \begin{bmatrix} -1, & 3/2 \\ 0 & 1/2 \end{bmatrix} \qquad C^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A_{B,C} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\rightarrow L(b_2) = L((3,4)) = (-4,-2) \rightarrow \ldots$

$$C^{-1}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A_{B,C}\,[v]_B = [L(v)]_C$$

application check:

$$v = \begin{bmatrix} 11 \\ 6 \end{bmatrix} \longrightarrow L(v) = \begin{bmatrix} -58 \\ -42 \end{bmatrix}$$

$L: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \longmapsto (-8x+5y, -6x+4y)$

$$[v]_B = B^{-1} v = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 11 \\ 6 \end{bmatrix} = \begin{bmatrix} -13 \\ 8 \end{bmatrix}$$

$$[L(v)]_C = C^{-1} . L(v) = \begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} -58 \\ -42 \end{bmatrix} = \begin{bmatrix} -5 \\ -21 \end{bmatrix}$$

58 − 63

$$A = \begin{bmatrix} -8 & 5 \\ -6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 5 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 11 \\ 6 \end{bmatrix} = \begin{bmatrix} -58 \\ -42 \end{bmatrix}$$

v in standart basis

$L(v)$ in the standard basis

$$A_{B,C}\,[v]_B \overset{?}{=} [L(v)]_C$$

$$\begin{bmatrix} -13 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} -5 \\ -21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -13 \\ 8 \end{bmatrix} = \begin{bmatrix} -5 \\ -21 \end{bmatrix}$$

$L: \mathbb{R}^2 \to \mathbb{R}^2$

$x b_1 + y b_2 \longmapsto (x+y)c_1 + (x-y)c_2$

$[v]_B \longmapsto [L(v)]_C$

$L: V \to W$

st. st.

✓ $A_{B,C} = \begin{bmatrix} [L(b_1)]_C & [L(b_2)]_C & \cdots \end{bmatrix}$

? ← $A_{B,E} = \begin{bmatrix} L(b_1) & L(b_2) & \cdots \end{bmatrix}$

? ← $A_{E,C} = \begin{bmatrix} [L(e_1)]_C & [L(e_2)]_C & \cdots \end{bmatrix}$

✓ $A_{E,E} = \begin{bmatrix} L(e_1) & L(e_2) \end{bmatrix}$

Ex $L: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (-8x+5y, -6x+4y)$

$$B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{b_2} \right\}$$

Find the rep. matrix $A_{B,E} = ?$ $E \to$ standart

$L(b_1) = L((1,2)) = (2,2)$

$L(b_1) = L((1,2)) = (2,2)$

$L(b_2) = L((3,4)) = (-4,-2)$

Find the rep. matrix $A_{B,\textcircled{E}} = ?$ $\to$ standart

$$A_{B,E} = \begin{bmatrix} | & | \\ L(b_1) & L(b_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$$

check

$$A_{B,E}\,[v]_B \overset{?}{=} L(v)$$

$$\begin{bmatrix} -58 \\ -42 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -13 \\ 8 \end{bmatrix} = \begin{bmatrix} -58 \\ -42 \end{bmatrix}$$

Ex $L : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (-8x+5y, -6x+4y)$

$C = \left\{ \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{c_1}, \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{c_2} \right\}$ (a basis for the RHS vector space) -2

Find the rep. matrix $A_{E,C} = ?$

$$A_{E,C} = \begin{bmatrix} | & | \\ [L(e_1)]_C & [L(e_2)]_C \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}$$

$L(e_1) = L((1,0)) = (-8,-6)$

$8-9$

$$\underbrace{\begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix}}_{C^{-1}}\begin{bmatrix} -8 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$L(e_2) = L((0,1)) = (5,4)$

$-5+6$

$$\begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

check

$$A_{E,C}\, v \overset{?}{=} [L(v)]_C$$

$$\begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ -21 \end{bmatrix} \checkmark$$

$-33+12$

4. Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{pmatrix}$$

Determine the standard matrix representation A of L, and use A to find $L(\mathbf{x})$ for each of the following vectors $\mathbf{x}$:

(a) $\mathbf{x} = (1,1,1)^T$ **(b)** $\mathbf{x} = (2,1,1)^T$

(c) $\mathbf{x} = (-5,3,2)^T$

$L : \mathbb{R}^3 \to \mathbb{R}^3$

$(x,y,z) \mapsto (2x-y-z, 2y-x-z, 2z-x-y)$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$L(e_1) = L((1,0,0)) = (2,-1,-1)$

$L(e_2) = L((0,1,0)) = (-1,2,-1)$

$L(e_3) = L((0,0,1)) = (-1,-1,2)$

b) $$\overset{A}{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \checkmark$$

$-2-1+2$

6. Let

$$B = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$L : \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \longmapsto x b_1 + y b_2 + (x+y) b_3$

$$B = \left\{ \mathbf{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix},\ \mathbf{b}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$\longrightarrow \quad L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3$$

$\longrightarrow$ Find the matrix A representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$A_{E,B} = ?$$

$$(x,y) \longmapsto x\,b_1 + y\,b_2 + (x+y)\,b_3$$

$$L: \ (x,y) \longrightarrow x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}1\\0\\1\end{bmatrix} + (x+y)\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$L: \ (x,y) \longmapsto (x+y,\ 2x+y,\ x+2y)$$

st. st.

$$A_{E,B} = \begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}$$

$$B = \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} \qquad \operatorname{adj}(B) = \begin{bmatrix}-1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & -1\end{bmatrix} \qquad B^{-1} = \begin{bmatrix}1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & 1/2\\ -1/2 & 1/2 & 1/2\end{bmatrix}$$

$$\det(B) = 1.(-1) - 1.1 + 0 = -2$$

$$L(e_1) = L\big((1,0)\big) = (1,2,1) \longrightarrow \underbrace{\begin{bmatrix}1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & 1/2\\ -1/2 & 1/2 & 1/2\end{bmatrix}}_{B^{-1}} \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\tfrac{1}{2} + 1 - \tfrac{1}{2}$$

$$L(e_2) = L\big((0,1)\big) = (1,1,2) \longrightarrow \begin{bmatrix}1/2 & 1/2 & -1/2\\ 1/2 & -1/2 & 1/2\\ -1/2 & 1/2 & 1/2\end{bmatrix} \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$-\tfrac{1}{2} + \tfrac{1}{2} + 1$$

18. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\mathcal{U} = \left\{ \mathbf{u}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix}1\\2\\1\end{bmatrix},\ \mathbf{u}_3 = \begin{bmatrix}-1\\1\\1\end{bmatrix} \right\}$$

and

$$B = \left\{ \mathbf{b}_1 = (1, -1)^T, \quad \mathbf{b}_2 = (2, -1)^T \right\}$$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases $\mathcal{U}$ and B: $\rightarrow A_{\mathcal{U},B} = ?$

(a) $L(\mathbf{x}) = (x_3, x_1)^T$

$\rightarrow$ (b) $L(\mathbf{x}) = (x_1 + x_2, x_1 - x_3)^T$

(c) $L(\mathbf{x}) = (2x_2, -x_1)^T$

$$B = \Big\{ \underbrace{\begin{bmatrix}1\\-1\end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{b_2} \Big\}$$

$$B = \begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix} \qquad -1-(-2) = 1$$

$$B^{-1} = \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}$$

b)

$$L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$(x,y,z) \longmapsto (x+y,\ x-z)$$

$$A_{\mathcal{U},B} = \begin{bmatrix} [L(u_1)]_B & [L(u_2)]_B & [L(u_3)]_B \end{bmatrix}$$

$$= \begin{bmatrix}-5 & -3 & 4\\ 3 & 3 & -2\end{bmatrix}$$

$$L(u_1) = L\big((1,0,-1)\big) = (1,2) \qquad \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-5\\3\end{bmatrix}$$

$$L(u_2) = L\big((1,2,1)\big) = (3,0) \qquad \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}-3\\3\end{bmatrix}$$

$$L(u_3) = L\big((-1,1,1)\big) = (0,-2) \qquad \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$$

application

$$2\underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{u_1} + 3\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}}_{u_2} - 1\underbrace{\begin{bmatrix}-1\\1\\1\end{bmatrix}}_{u_3} = \begin{bmatrix}6\\5\\0\end{bmatrix}$$

$$[v]_{\mathcal{U}} = \begin{bmatrix}2\\3\\-1\end{bmatrix}$$

$$v = \begin{bmatrix}6\\5\\0\end{bmatrix} \xrightarrow[(x,y,z) \mapsto (x+y,\, x-z)]{} L(v) = \begin{bmatrix}11\\6\end{bmatrix}$$

$$[L(v)]_B = \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}11\\6\end{bmatrix} = \begin{bmatrix}-23\\17\end{bmatrix}$$

$$A_{\mathcal{U},\mathcal{B}}\,[v]_{\mathcal{U}} \stackrel{?}{=} [L(v)]_{\mathcal{B}}$$

$$\begin{bmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}_{3\times 1} = \begin{bmatrix} -23 \\ 17 \end{bmatrix}_{2\times 1}$$

a)